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A (2+1)-dimensional derivative Toda equation in the context of the Kaup-Newell spectral problem

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Abstract

A (2+1)-dimensional derivative Toda equation is derived from the Lax triad composed of the Kaup–Newell, the negative Kaup–Newell and a discrete spectral problem. Two integrable Hamiltonian systems and an integrable symplectic map are derived from the Lax triad. They are straightened out in the Jacobi variety of the associated hyperelliptic curve. A finite genus solution is obtained through a technique based on the Riemann–Jacobi inversion theorem. Besides, explicit solutions are calculated for other associated integrable models, including the mKP equation and the nKN equation.

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1. Introduction

In the present paper, finite genus solutions [1, 2] of integrable equations associated with the Kaup–Newell spectral problem (KN) [3] are investigated, with special emphasis on the (2+1)-dimensional derivative Toda equation (dToda):

$$\frac{\partial^2\varphi_n}{\partial x\partial y} + (e^{\varphi_{n+1}-\varphi_n} - e^{\varphi_n-\varphi_{n-1}})\frac{\partial\varphi_n}{\partial x} = 0, \tag{1.1}$$

similar to the traditional (2+1) Toda equation [4]: $\varphi_{n,xy} = e^{\varphi_{n+1}-\varphi_n} - e^{\varphi_n-\varphi_{n-1}}$, but with different integrable structure. Equation (1.1) is derived from the Lax triad composed of the Kaup–Newell, the negative Kaup–Newell (nKN) and a discrete eigenvalue problem:

$$\partial_x\chi = V_1\chi, \qquad V_1 = \begin{pmatrix} \lambda^2/2 & u\lambda \\ v\lambda & -\lambda^2/2 \end{pmatrix}, \tag{1.2}$$

$$\partial_y\chi = V_{-1}\chi, \qquad V_{-1} = \begin{pmatrix} -\lambda^{-2}/2 + rs & -r\lambda^{-1} \\ -s\lambda^{-1} & \lambda^{-2}/2 - rs \end{pmatrix}, \tag{1.3}$$

$$E\chi = U\chi, \qquad U = \begin{pmatrix} 0 & -a^{-1} \\ a & -b\lambda - \lambda^{-1} \end{pmatrix}, \tag{1.4}$$

where $\partial_x = \partial/\partial x$ and E is the shift operator: $(Ef)(n) = f(n+1)$, $(E^- f)(n) = f(n-1)$ for the scalar-valued function f and $E\chi = E(\chi^1, \chi^2)^T = (E\chi^1, E\chi^2)^T$ for the vector-valued function χ.

Each spectral problem has independent interest. Other integrable partial differential equations are derived from them. For example, the first isospectral equation for KN is the derivative nonlinear Schrödinger equation [3–6], while that for nKN reads (see proposition 2.3–4 below)

$$\begin{cases} r_{\tau_{-2}} = -r_{yy} + 2r^2 s_y + 2r^3 s^2, \\ s_{\tau_{-2}} = +s_{yy} + 2s^2 r_y - 2r^2 s^3. \end{cases} \tag{1.5}$$

It is well known that the Kadomtsev–Petviashvili equation (KP) can be derived from the first two ZS-AKNS equations as a compatible condition [7, 8]. Similarly, both KN and nKN yield the same mKP equation [9–13], though in different arguments (see propositions 2.2 and 2.4 below):

$$w_{\tau_3} = \tfrac{1}{4}(w_{xx} - 2w^3)_x + \tfrac{3}{4}\partial_x^{-1} w_{\tau_2\tau_2} - \tfrac{3}{2} w_x \partial_x^{-1} w_{\tau_2}, \tag{1.6}$$

$$\widetilde{w}_{\tau_{-3}} = \tfrac{1}{4}(\widetilde{w}_{yy} - 2\widetilde{w}^3)_y + \tfrac{3}{4}\partial_y^{-1} \widetilde{w}_{\tau_{-2}\tau_{-2}} - \tfrac{3}{2} \widetilde{w}_y \partial_y^{-1} \widetilde{w}_{\tau_{-2}}. \tag{1.7}$$

In 1978, Moser first used the generating function in the study of integrable reduction of the KdV equation to a mechanical problem of Neumann [14, 15]. A powerful tool, the Lax–Moser matrix, is developed from his method to calculate the explicit solution for the KP equation [16] and some other (2+1)-dimensional integrable models containing both continuous and discrete arguments [17–19].

The Lax–Moser matrix (see equation (3.1) below) is obtained through the nonlinearization of the KN spectral problem. It determines two objects: (i) *the integrals* $\{H_k\}$, which are the coefficients in the power series expansions of the square root of its determinant (see equation (3.7) below); (ii) *the algebraic curve* Γ, whose affine equation coincides with the characteristic equation of the Lax–Moser matrix up to a polynomial factor (see equation (3.16) below).

It turns out that the canonical equations for the Hamiltonian H_1, H_{-1} are exactly the KN and the nKN equations (1.2) and (1.3) under the Bargmanns constraint $(u, v) = f^+(p, q), (r, s) = f^-(p, q)$, given by equations (3.12) and (3.14), respectively. On the other hand, equation (1.4) becomes an integrable symplectic map S under another constraint $(a, b) = f_S(p, q)$, given by equation (4.2), sharing the same integrals.

Finite genus solutions are obtained in three steps.

(i) *Decomposition*. Special solution of PDE is reduced into compatible solution of ODEs as

$$(2+1)\ \mathrm{d\,Toda}(1.1): \quad (H_1), (H_{-1}), S; \tag{1.8}$$

$$nKN(1.5): \quad (H_{-1}), (H_{-2}); \tag{1.9}$$

$$mKP(1.6): \quad (H_1), (H_2), (H_3); \tag{1.10}$$

$$mKP(1.7): \quad (H_{-1}), (H_{-2}), (H_{-3}). \tag{1.11}$$

(ii) *Straightening out*. The flows generated by H_k, H_{-k} and S are linearized in the Jacobi variety of Γ with constant evolution speeds Ω_k, Ω_{-k} and Ω_S, respectively. The Abel–Jacobi solutions are linear combinations of these angle velocities.

(iii) *Inversion*. The final form of solution is obtained through two substeps:

$$\text{Abel–Jacobi variables} \xrightarrow{(a)} \text{elliptic variables} \xrightarrow{(b)} \text{potential variables.}$$

The substep (a) is guaranteed by the Jacobi inversion theorem and calculated through the trace formulae. The substep (b) is sometimes rather troublesome to get satisfied results. Tedious calculations are concerned to discuss the nature of constants, which appeared in the process of integration.

2. Integrable PDEs associated with KN and nKN

In this section, various integrable partial differential equations associated with the KN and nKN spectral problems are presented in the zero-curvature forms, which are compatible conditions of two linear spectral problems (the Lax pair) in the (1+1)-dimensional case, or that of the Lax triad in the (2+1) case. These will lead to the decomposition into finite-dimensional integrable systems and integrable symplectic map, as will be shown in the next sections.

(i) *The KN hierarchy*

The deduction of the Kaup–Newell hierarchy is provided by the fundamental identity for any smooth function $\gamma : \mathbb{R} \to \mathbb{R}^3$ with $V = \sigma_\lambda(\gamma)$:

$$\partial_x V - [V_1, V] = \sigma_\lambda\{(K^+ - \lambda^2 J^+)\gamma\}, \tag{2.1}$$

$$\sigma_\lambda = \begin{pmatrix} \lambda\gamma^3 & \gamma^1 \\ \gamma^2 & -\lambda\gamma^3 \end{pmatrix}, \qquad K^+ = \begin{pmatrix} \partial_x & 0 & 0 \\ 0 & \partial_x & 0 \\ v & -u & \partial_x \end{pmatrix}, \qquad J^+ = \begin{pmatrix} 1 & 0 & -2u \\ 0 & -1 & 2v \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.2}$$

The Lenart sequence [20] $\{g_j\}$ are defined recursively by

$$J^+ g_0 = 0, \qquad K^+ g_k = J^+ g_{k+1}, \qquad (k = 1, 2, \ldots), \tag{2.3}$$

$$g_0 = \begin{pmatrix} u \\ v \\ 1/2 \end{pmatrix}, \qquad g_1 = \begin{pmatrix} u_x - 2u^2 v \\ -v_x - 2uv^2 \\ -uv \end{pmatrix}, \qquad g_2 = \begin{pmatrix} u_{xx} - 6uvu_x + 6u^3v^2 \\ v_{xx} + 6uvv_x + 6u^2v^3 \\ -u_x v + 6uv_x + 3u^2v^2 \end{pmatrix}. \tag{2.4}$$

Obviously $V_1 = \sigma_\lambda(g_0\lambda)$. Define $V_k = \sigma_\lambda(\gamma_k)$, $\gamma_k = \sum_{j=0}^{k-1} g_j \lambda^{2k-2j-1}$. Then we have a direct relation to establish the equivalence between the zero-curvature equation and the isospectral equation

$$\partial_{\tau_k} V_1 - \partial_x V_k + [V_1, V_k] = \lambda\sigma_\lambda\left\{(u, v, 0)^T_{\tau_k} - J^+ g_k\right\}.$$

Proposition 2.1. *V_1, V_k are the Lax pair for the kth KN equation*

$$\partial_{\tau_k}\begin{pmatrix} u \\ v \end{pmatrix} = X_k \triangleq \begin{pmatrix} (J^+g_k)^1 \\ (J^+g_k)^2 \end{pmatrix}, \tag{2.5}$$

Proposition 2.2. *Let $u(x, \tau_2, \tau_3)$, $v(x, \tau_2, \tau_3)$ be a compatible solution of*

$$\partial_{\tau_2}\begin{pmatrix} u \\ v \end{pmatrix} = X_2 = \begin{pmatrix} u_{xx} - 2(u^2v)_x \\ -v_{xx} - 2(uv^2)_x \end{pmatrix}, \qquad \partial_{\tau_3}\begin{pmatrix} u \\ v \end{pmatrix} = X_3 = \begin{pmatrix} (u_{xx} - 6uvu_x + 6u^3v^2)_x \\ (v_{xx} + 6uvv_x + 6u^2v^3)_x \end{pmatrix} \tag{2.6}$$

Then $w = uv$ solves the mKP equation (1.6).

(ii) *The nKN hierarchy*

The deduction of the isospectral hierarchy of equation (1.3) is based on another fundamental identity with $V = \sigma_\lambda^-(\gamma)$:

$$\partial_y V - [V_{-1}, V] = \hat{\sigma}_\lambda^-\{(K^- - \lambda^{-2} J^-)\gamma\}, \tag{2.7}$$

$$\begin{aligned} \sigma_\lambda^-(\gamma) &= \begin{pmatrix} -\gamma^3\lambda^{-2} + s\gamma^1 + r\gamma^2 & -\gamma^1\lambda^{-1} \\ -\gamma^2\lambda^{-1} & \gamma^3\lambda^{-2} - s\gamma^1 - r\gamma^2 \end{pmatrix}, \\ \hat{\sigma}_\lambda^-(\xi) &= \begin{pmatrix} \xi^3 + s\xi^1 + r\xi^2 & -\xi^1\lambda^{-1} \\ -\xi^2\lambda^{-1} & -\xi^3 - s\xi^1 - r\xi^2 \end{pmatrix}, \\ K^- &= \begin{pmatrix} \partial_y & 2r^2 & 0 \\ -2s^2 & \partial_y & 0 \\ s_y + 2rs^2 & r_y - 2r^2 s & 0 \end{pmatrix}, \qquad J^- = \begin{pmatrix} -1 & 0 & 2r \\ 0 & 1 & -2s \\ 0 & 0 & \partial_y \end{pmatrix}. \end{aligned} \tag{2.8}$$

The Lenart sequence $\{h_{-k}\}$ are defined recursively by

$$J^- h_0 = 0, \qquad J^- h_{-k-1} = K^- h_{-k}, \qquad (k = 0, 1, 2, \ldots), \tag{2.9}$$

$$\begin{aligned} h_0 &= \begin{pmatrix} r \\ s \\ 1/2 \end{pmatrix}, \qquad h_{-1} = \begin{pmatrix} -r_y \\ s_y \\ rs \end{pmatrix}, \qquad h_{-2} = \begin{pmatrix} r_{yy} - 2r^2 s_y - 2r^3 s^2 \\ s_{yy} + 2s^2 r_y - 2r^2 s^3 \\ -r^2 s^2 \end{pmatrix}, \\ h_{-3} &= \begin{pmatrix} -r_{yyy} + 6r r_y s_y + 6r^2 s^2 r_y \\ s_{yyy} + 6s r_y s_y - 6r^2 s^2 s_y \\ r_y s_y + 2rs^2 r_y - 2r^2 s s_y - 2r^3 s^3 \end{pmatrix}. \end{aligned} \tag{2.10}$$

Define $V_{-k} = \sigma_\lambda^-(\gamma_{-k})$, where

$$\gamma_{-k} \triangleq h_0 \lambda^{-2k+2} + h_{-1}\lambda^{-2k+4} + \cdots + h_{-k+1} + \left(0, 0, h_{-k}^3\right)^T \lambda^2.$$

It is little different that there is an extra term. The corresponding equality reads

$$\partial_{\tau_{-k}} V_{-1} - \partial_y V_{-k} + [V_{-1}, V_{-k}] = \hat{\sigma}_\lambda^-\left\{(r, s, 0)_{\tau_{-k}}^T - \left(-h_{-k}^1, h_{-k}^2, 0\right)^T\right\}.$$

Proposition 2.3. *V_{-1}, V_{-k} are the Lax pair for the kth nKN equation:*

$$\partial_{\tau_{-k}} \begin{pmatrix} r \\ s \end{pmatrix} = X_{-k} \triangleq \begin{pmatrix} -h_{-k}^1 \\ h_{-k}^2 \end{pmatrix}, \tag{2.11}$$

Proposition 2.4. *Let r, s be a compatible solution of (X_{-2}) and (X_{-3})*

$$\begin{pmatrix} r \\ s \end{pmatrix}_{\tau_{-2}} = \begin{pmatrix} -r_{yy} + 2r^2 s_y + 2r^3 s^2 \\ s_{yy} + 2s^2 r_y - 2r^2 s^3 \end{pmatrix}, \qquad \begin{pmatrix} r \\ s \end{pmatrix}_{\tau_{-3}} = \begin{pmatrix} r_{yyy} - 6r r_y s_y - 6r^2 s^2 r_y \\ s_{yyy} + 6s r_y s_y - 6r^2 s^2 s_y \end{pmatrix} \tag{2.12}$$

Then $\widetilde{w} = rs$ solves the mKP equation (1.7).

Remark. Since $X_1 = (u_x, v_x)^T$, $X_{-1} = (r_y, s_y)^T$, the notations $x = \tau_1$, $y = \tau_{-1}$ are reasonable.

Proposition 2.5. *Let $\partial_x r = -u$, $\partial_x s = v$. Then $\partial_y V_1 - \partial_x V_{-1} + [V_1, V_{-1}] = 0$ if and only if*

$$\begin{cases} r_{xy} - 2rsr_x + r = 0, \\ s_{xy} + 2rss_x + s = 0. \end{cases} \tag{2.13}$$

Among the integrable equations associated with KN, equation (2.13) has a special position, since it has KN and nKN as its Lax pair. A compatible solution of (H_1) and (H_{-1}) leads to the finite genus solution in the Abel–Jacobi coordinates (see equation (6.8) below). Nevertheless, essential difficulties are encountered in the inversion process. An explicit solution in the potential variables r, s has not been obtained.

(iii) *Integrable equations containing discrete variable*

We write $a(x, n) = a_n(x)$, etc for short. Through direct calculations we have

Proposition 2.6. *Let* $u = -1/(a_{n-1}b_{n-1})$, $v = a_n/b_n$. *Then* $\partial_x U = (EV_1)U - UV_1$ *if and only if*

$$\partial_x a_n = \frac{a_n}{b_n}, \qquad \partial_x b_n = b_n\left(\frac{a_{n+1}}{a_n b_{n+1} b_n} - \frac{a_n}{a_{n-1} b_n b_{n-1}}\right). \tag{2.14}$$

Proposition 2.7. *Let* $r = -1/a_{n-1}$, $s = a_n$. *Then* $\partial_y U = (EV_{-1})U - UV_{-1}$ *if and only if*

$$\partial_y a_n = a_n\left(\frac{a_{n+1}}{a_n} + \frac{a_n}{a_{n-1}} - b_n\right), \qquad \partial_y b_n = b_n\left(\frac{a_{n+1}}{a_n} - \frac{a_n}{a_{n-1}}\right). \tag{2.15}$$

Proposition 2.8. *Let* $(u, v), (r, s)$ *be a compatible solution of equation (2.14)–(2.15). Then* $\varphi_n = \ln a_n$ *solves the (2+1) d-Toda equation (1.1).*

Proposition 2.9. *Let* $(u, v), (r, s)$ *be given as in proposition 2.6–2.7. Then* $\varphi_n(x, y) = \ln a_n(x, y)$, *given by the compatible solution of equation (1.2)–(1.4), solves the (2+1) d-Toda equation (1.1).*

3. The Liouville-integrable system (H_k) and (H_{-k})

(i) *The Lax–Moser matrix*

The Lax–Moser matrix is calculated by using the fundamental identity (2.1), in a similar way as in [16–19]. To omit the lengthy argument, we begin with direct verification of the results. Let $A = \mathrm{diag}(\alpha_1, \ldots, \alpha_N)$, whose diagonal elements are distinct non-zero constants. Denote

$$\langle \xi, \eta \rangle = \sum_{j=1}^{N} \xi_j \eta_j, \qquad Q_\lambda(\xi, \eta) = \langle (\lambda^2 - A^2)^{-1} \xi, \eta \rangle.$$

Define the Lax–Moser matrix:

$$V_\lambda = \begin{pmatrix} 1/2 + Q_\lambda(A^2 p, q) & -\lambda Q_\lambda(Ap, p) \\ \lambda Q_\lambda(Aq, q) & -1/2 - Q_\lambda(A^2 p, q) \end{pmatrix}. \tag{3.1}$$

Consider the canonical equations with the Hamiltonian $F_\lambda = \det V_\lambda$

$$F_\lambda = -[1/2 + Q_\lambda(A^2 p, q)]^2 + \lambda^2 Q_\lambda(Ap, p) Q_\lambda(Aq, q), \tag{3.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t_\lambda}\begin{pmatrix} p_j \\ q_j \end{pmatrix} = \begin{pmatrix} -\partial F_\lambda/\partial q_j \\ \partial F_\lambda/\partial p_j \end{pmatrix} = W(\lambda, \alpha_j)\begin{pmatrix} p_j \\ q_j \end{pmatrix}, \tag{3.3}$$

$$W(\lambda, \mu) = \frac{2\mu}{\lambda^2 - \mu^2}\begin{pmatrix} \mu V_\lambda^{11} & \lambda V_\lambda^{12} \\ \lambda V_\lambda^{21} & -\mu V_\lambda^{11} \end{pmatrix}. \tag{3.4}$$

Through direct calculations we obtain the Lax equation along the F_λ flow satisfied by V_μ:

$$\frac{\mathrm{d}V_\mu}{\mathrm{d}t_\lambda} = [W(\lambda, \mu), V_\mu]. \tag{3.5}$$

As a general fact for the Lax equation, $F_\mu = \det V_\mu$ is constant along the flow. Thus we obtain Moser's formula:

$$(F_\mu, F_\lambda) = 0, \qquad \forall \mu, \lambda \in \mathbb{C}, \tag{3.6}$$

since the derivative of a smooth function along the Hamiltonian flow is equal to its Poisson bracket with the Hamiltonian. It turns out that a more essential role is played by the square root H_λ, defined as

$$-4F_\lambda = (-4H_\lambda)^2. \tag{3.7}$$

(ii) *The integrals*

Integrals are determined through power series expansions

$$F_\lambda = -\frac{1}{4} + \sum_{j=1}^{\infty} F_j \lambda^{-2j}, \qquad H_\lambda = -\frac{1}{4} + \sum_{j=1}^{\infty} H_j \lambda^{-2j},$$
$$F_\lambda = \sum_{k=0}^{\infty} F_{-k} \lambda^{2k}, \qquad H_\lambda = \sum_{k=0}^{\infty} H_{-k} \lambda^{2k},$$

for $|\lambda| > \max\{|\alpha_1|, \ldots, |\alpha_N|\}$, $|\lambda| < \min\{|\alpha_1|, \ldots, |\alpha_N|\}$, respectively. The explicit formula for $F_{\pm k}$ and recursive formula for $H_{\pm k}$ are as follows:

$$F_k = -\langle A^{2k} p, q\rangle - \sum_{i+j=k; i,j\geqslant 1} \langle A^{2i} p, q\rangle\langle A^{2j} p, q\rangle - \sum_{i+j=k+1; i,j\geqslant 1} \langle A^{2i-1} p, p\rangle\langle A^{2j-1} q, q\rangle;$$

$$F_{-k} = -\langle A^{-2k} p, q\rangle - \sum_{i+j=k; i,j\geqslant 0} \langle A^{-2i} p, q\rangle\langle A^{-2j} p, q\rangle + \sum_{i+j=k-1; i,j\geqslant 0} \langle A^{-2i-1} p, p\rangle\langle A^{-2j-1} q, q\rangle;$$

$$H_k = \frac{1}{2} F_k + 2 \sum_{i+j=k; i,j\geqslant 1} H_i H_j;$$

$$H_{-k} = -\frac{F_{-k}}{8H_0} - \frac{1}{2H_0} \sum_{i+j=k; i,j\geqslant 1} H_{-i} H_{-j}.$$

with the first few members:

$$\begin{aligned}
F_1 &= -\langle A^2 p, q\rangle + \langle Ap, p\rangle\langle Aq, q\rangle, \\
F_0 &= -(2\langle p, q\rangle - 1)^2, \\
F_1 &= -\left(\langle p, q\rangle - \frac{1}{2}\right)\langle A^{-2} p, q\rangle + \langle A^{-1} p, p\rangle\langle A^{-1} q, q\rangle; \\
H_1 &= -\frac{1}{2}\langle A^2 p, q\rangle + \frac{1}{2}\langle Ap, p\rangle\langle Aq, q\rangle, \\
H_0 &= \frac{1}{4}(2\langle p, q\rangle - 1), \\
H_{-1} &= -\frac{1}{2}\langle A^{-2} p, q\rangle - \frac{1}{2}\frac{\langle A^{-1} p, p\rangle\langle A^{-1} q, q\rangle}{\left(\langle p, q\rangle - \frac{1}{2}\right)^2}.
\end{aligned} \tag{3.8}$$

By the Leibniz rule of the Poisson bracket, from equations (3.6) and (3.7) and their expansions we have

$$\begin{aligned}
&(F_\lambda, F_\mu) = (F_\lambda, H_\mu) = (H_\lambda, H_\mu) = 0, \qquad \forall \lambda, \mu \in \mathbb{C}; \\
&(F_i, F_j) = (F_i, H_j) = (H_i, H_j) = 0, \qquad \forall i, j \in \mathbb{Z}
\end{aligned} \tag{3.9}$$

The Hamiltonian system $(\mathbb{R}^{2N}, dp \wedge dq, F_k)$ with the variable t_k is denoted by (F_k) for short. Similarly we have (F_{-k}), (H_k), (H_{-k}), (H_λ) with variables $t_{-k}, \tau_k, \tau_{-k}, \tau_\lambda$, respectively. The Leibniz rule of the Poisson bracket gives rise to

$$\frac{\mathrm{d}}{\mathrm{d}t_\lambda} = -8H_\lambda \frac{\mathrm{d}}{\mathrm{d}\tau_\lambda}. \tag{3.10}$$

The canonical equation of the Hamiltonian system (H_1), (H_{-1}) are put in the form

$$\partial_x \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -\partial H_1/\partial q \\ \partial H_1/\partial p \end{pmatrix} = \begin{pmatrix} A^2/2 & uA \\ vA & -A^2/2 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}, \tag{3.11}$$

$$(u, v) = f^+(p, q) = (-\langle Ap, p\rangle, \langle Aq, q\rangle), \tag{3.12}$$

$$\partial_y \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -\partial H_{-1}/\partial q \\ \partial H_{-1}/\partial p \end{pmatrix} = \begin{pmatrix} -A^{-2}/2 + rs & -rA^{-1} \\ -sA^{-1} & A^{-2}/2 - rs \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}, \tag{3.13}$$

$$(r, s) = f^-(p, q) = \frac{1}{2\langle p, q\rangle - 1}(-\langle A^{-1}p, p\rangle, \langle A^{-1}q, q\rangle), \tag{3.14}$$

respectively. The jth component of equation (3.11) reads

$$\partial_x \begin{pmatrix} p_j \\ q_j \end{pmatrix} = \begin{pmatrix} \alpha_j^2/2 & u\alpha_j \\ v\alpha_j & -\alpha_j^2/2 \end{pmatrix} \begin{pmatrix} p_j \\ q_j \end{pmatrix}.$$

It is exactly the KN equation (1.2) with $\lambda = \alpha_j, \chi = (p_j, q_j)^T$ and u, v expressed as in equation (3.12). Similar relation exists between (H_{-1}), given by equation (3.13) and nKN (1.3).

(iii) *The algebraic curve*

By equation (3.2), F_λ is a rational function of $\zeta = \lambda^2$ with single poles at $\zeta = \alpha_j^2$, $1 \leqslant j \leqslant N$. Thus

$$F_\lambda = -\frac{b(\zeta)}{4\alpha(\zeta)} = -\frac{\prod_1^N \left(\zeta - b_j^2\right)}{4\prod_1^N \left(\zeta - \alpha_j^2\right)} = -\frac{R(\zeta)}{4\alpha^2(\zeta)}. \tag{3.15}$$

Define an algebraic curve,

$$\Gamma: \quad \xi^2 = R(\zeta) \triangleq \prod_{k=1}^{2N} \left(\zeta - \lambda_k^2\right), \tag{3.16}$$

where $\lambda_j = \alpha_j, \lambda_{N+j} = b_j, 1 \leqslant j \leqslant N, R(\zeta) = \alpha(\zeta)b(\zeta)$. Γ has genus $g = N - 1$. Corresponding to each $\zeta \in \mathbb{C}$, there are two points on Γ: $P(\zeta)$ and $P^-(\zeta)$, with ξ equals to $\pm\sqrt{R(\zeta)}$, respectively. In particular, we have $0_l : \zeta = 0, \xi = (-1)^l\sqrt{R(0)}$. The affine equation of Γ near ∞ is

$$\eta^2 = R_*(z) \triangleq \prod_{k=1}^{2N} \left(1 - \lambda_k^2 z\right), \tag{3.17}$$

with $z = \zeta^{-1}, \eta = \zeta^{-N}\xi$. Similarly we have two infinities $\infty_l : z = 0, \eta = (-1)^l$. The functions F_λ, H_λ and equation (3.10) are represented as

$$-4F_\lambda = \frac{R(\zeta)}{\alpha^2(\zeta)}, \qquad -4H_\lambda = \frac{\sqrt{R(\zeta)}}{\alpha(\zeta)}, \qquad \frac{\mathrm{d}}{\mathrm{d}t_\lambda} = \frac{\sqrt{R(\zeta)}}{2\alpha(\zeta)} \frac{\mathrm{d}}{\mathrm{d}\tau_\lambda}. \tag{3.18}$$

Let $P_0 \in \Gamma$ be fixed. Introduce the quasi-Abel–Jacobi variables $(i = 1, \ldots, g)$

$$\widetilde{\phi}_i = \sum_{k=1}^{g} \int_{P_0}^{P(\mu_k^2)} \widetilde{\omega}_i, \qquad \widetilde{\omega}_i = \frac{\zeta^{g-i}}{2\sqrt{R(\zeta)}} \mathrm{d}\zeta, \tag{3.19}$$

where $\widetilde{\omega}_1, \ldots, \widetilde{\omega}_g$ constitute the basis of holomorphic differentials of Γ. The elliptic variables μ_k^2, ν_k^2 are defined as the zeros of $V_\lambda^{12}, V_\lambda^{21}$, given by the components of equation (3.1), with the factorizations

$$\begin{aligned} V_\lambda^{12} &= -\lambda\langle Ap, p\rangle \frac{\widetilde{m}(\zeta)}{\alpha(\zeta)}, \qquad \widetilde{m}(\zeta) = \prod_{j=1}^{g}\left(\zeta - \mu_j^2\right); \\ V_\lambda^{21} &= \lambda\langle Aq, q\rangle \frac{\widetilde{n}(\zeta)}{\alpha(\zeta)}, \qquad \widetilde{n}(\zeta) = \prod_{j=1}^{g}\left(\zeta - \nu_j^2\right). \end{aligned} \tag{3.20}$$

Due to lemma (4.3) below, we need only deal with $\left\{\mu_k^2\right\}$. Put $\lambda = \mu_k$ $\left(\zeta = \mu_k^2\right)$ into the equation

$$\frac{R(\zeta)}{4\alpha^2(\zeta)} = -F_\lambda = \left(V_\lambda^{11}\right)^2 + V_\lambda^{12}V_\lambda^{21}.$$

We get

$$V_\lambda^{11}\big|_{\lambda=\mu_k} = \frac{\sqrt{R\left(\mu_k^2\right)}}{2\alpha\left(\mu_k^2\right)}.$$

Consider the component equation for V_μ^{12} in equation (3.5)

$$\frac{\mathrm{d}V_\mu^{12}}{\mathrm{d}t_\lambda} = 2\left(W_{\lambda\mu}^{11}V_\mu^{12} - W_{\lambda\mu}^{12}V_\mu^{11}\right) = \frac{4\mu}{\lambda^2 - \mu^2}\left(\mu V_\lambda^{11}V_\mu^{12} - \lambda V_\lambda^{12}V_\mu^{11}\right).$$

After substituting V_μ, V_λ by equation (3.20), and putting $\mu = \mu_k$, we obtain

$$\frac{1}{2\sqrt{R\left(\mu_k^2\right)}}\frac{\mathrm{d}\left(\mu_k^2\right)}{\mathrm{d}t_\lambda} = \frac{\zeta}{\alpha(\zeta)}\frac{\widetilde{m}(\zeta)}{\left(\zeta - \mu_k^2\right)\widetilde{m}'\left(\mu_k^2\right)}. \tag{3.21}$$

Using the interpolation formula for the polynomial $\widetilde{m}(\zeta)$, we have $(i = 1, \ldots, g)$

$$\sum_{k=1}^{g}\frac{\left(\mu_k^2\right)^{g-i}}{2\sqrt{R\left(\mu_k^2\right)}}\frac{\mathrm{d}\left(\mu_k^2\right)}{\mathrm{d}t_\lambda} = \frac{\zeta}{\alpha(\zeta)}\sum_{k=1}^{g}\frac{\left(\mu_k^2\right)^{g-i}\widetilde{m}(\zeta)}{\left(\zeta - \mu_k^2\right)\widetilde{m}'\left(\mu_k^2\right)} = \frac{\zeta^{g-i+1}}{\alpha(\zeta)}.$$

By equation (3.19), finally we get

$$\frac{\mathrm{d}\widetilde{\phi}_i}{\mathrm{d}t_\lambda} = (\widetilde{\phi}_i, F_\lambda) = \frac{\zeta^{g-i+1}}{\alpha(\zeta)}, \qquad (i = 1, \ldots, g). \tag{3.22}$$

Proposition 3.1. *The quasi-Abel–Jacobi variables straighten out the F_k flow*

$$\frac{\mathrm{d}\widetilde{\phi}_i}{\mathrm{d}t_k} = (\widetilde{\phi}_i, F_k) = A_{k-i}, \tag{3.23}$$

$(1 \leqslant i \leqslant g, k = 1, 2, \ldots)$, *where A_j, A_{-j} are determined by* $(A_0 = 1)$*:*

$$\prod_{j=1}^{N}\frac{1}{1 - \alpha_j^2\zeta^{-1}} = \sum_{j=0}^{\infty}A_j\zeta^{-j}; \qquad A_{-j} = 0, \qquad (j = 1, 2, \ldots). \tag{3.24}$$

Lemma 3.2. *$F_0, F_1, \ldots, F_{N-1}$ are functionally independent in the open set $\{H_0 \neq 0\}$.*

Proof. Suppose $\sum_0^{N-1} c_k dF_k = 0$. According to the relation between the Poisson bracket and the symplectic structure $\omega^2 = dp \wedge dq$ [21], we have

$$0 = \omega^2 \left(I \sum_0^g c_k dF_k, Id\widetilde{\phi}_i \right) = \sum_0^g c_k \omega^2 (IdF_k, Id\widetilde{\phi}_i)$$
$$= \sum_0^g c_k (\widetilde{\phi}_i, F_k) = \sum_1^g c_k (\widetilde{\phi}_i, F_k),$$

where $(\widetilde{\phi}_i, F_0) = A_{-j} = 0$ is used. The coefficient matrix in the right-hand side is non-degenerate since it is triangular with diagonal element 1 by equation (3.23)–(3.24). Thus $c_1 = \cdots = c_{N-1} = 0$. We have $c_0 dF_0 = 0$, which implies $c_0 = 0$. □

Proposition 3.3. *Each of the Hamiltonian systems* $(F_\lambda), (H_\lambda), (F_k), (F_{-k}), (H_k), (H_{-k})$ *is completely integrable in the Liouville sense [21].*

(iv) *Straightening out of the* H_k*- and* H_{-k}*-flow*

Let $a_1, \ldots, a_g, b_1, \ldots, b_g$ be the normalized basis of homological cycles of Γ. Put $\widetilde{\omega} = (\widetilde{\omega}_1, \ldots, \widetilde{\omega}_g)^T$ and transform it into

$$\omega = C\widetilde{\omega} = (C_1\zeta^{g-1} + C_2\zeta^{g-2} + \cdots + C_g)\frac{\mathrm{d}\zeta}{2\sqrt{R(\zeta)}}. \tag{3.25}$$

Here C_j is the column vector $C = (A_{jk})^{-1}_{g\times g}$, where A_{jk} is the integral of $\widetilde{\omega}_j$ along a_k. The periodic vector δ_k, B_k are the integrals of ω along a_k, b_k, respectively. They span the lattice $\mathcal{T}$ in $\mathbb{C}^g$. Let B be the matrix with the column vector B_k. It is used to construct the theta function $\theta(\zeta, B)$ of the curve Γ [22, 23]. The Abel map $\mathcal{A}(P)$ is defined as the integral of ω from P_0 to P, with linear extension to the divisor group $\mathrm{Div}(\Gamma)$. The Abel–Jacobi variable is defined as

$$\phi = C\widetilde{\phi} = \mathcal{A}\left(\sum_{j=1}^g P\left(\mu_j^2\right)\right). \tag{3.26}$$

Change $\mathrm{d}/\mathrm{d}t_\lambda$ into $\mathrm{d}/\mathrm{d}\tau_\lambda$ by equation (3.18). Multiplied by C from the left-hand side, equation (3.22) becomes

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau_\lambda} = (\phi, H_\lambda) = \frac{1}{2\sqrt{R(\zeta)}}(C_1\zeta^g + C_2\zeta^{g-1} + \cdots + C_g\zeta). \tag{3.27}$$

Expand in the powers of $z = \zeta^{-1}, \zeta$, respectively. We have

Proposition 3.4. *The Abel–Jacobi variables* ϕ *straighten out the* H_k*- and* H_{-k}*-flow*

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau_k} = (\phi, H_k) = \Omega_k, \qquad \frac{\mathrm{d}\phi}{\mathrm{d}\tau_{-k}} = (\phi, H_{-k}) = \Omega_{-k}, \qquad \frac{\mathrm{d}\phi}{\mathrm{d}\tau_0} = 0, \tag{3.28}$$

$(k = 1, 2, \ldots)$, *where* Ω_k, Ω_{-k} *are determined by the expansions of the normalized basis* ω *of holomorphic differentials in the neighborhood of* $\infty_l, 0_l$, *respectively* $(l = 1, 2)$, $(\lambda^2 = \zeta = z^{-1})$:

$$\omega = (-1)^{l-1}\sum_{k=1}^{\infty}\Omega_k z^{k-1}\,\mathrm{d}z, \qquad \Omega_1 = \frac{1}{2}C_1, \tag{3.29}$$

$$\omega = (-1)^l \sum_{k=1}^{\infty}\Omega_{-k}\zeta^{k-1}\,\mathrm{d}\zeta, \qquad \Omega_{-1} = \frac{1}{2\sqrt{R(0)}}C_g. \tag{3.30}$$

4. The Liouville-integrable symplectic map S

Define a map $S : \mathbb{R}^{2N} \longrightarrow \mathbb{R}^{2N}$ by

$$\begin{pmatrix}\overline{p}\\ \overline{q}\end{pmatrix} = E\begin{pmatrix}p\\ q\end{pmatrix} = S(p,q) = \begin{pmatrix}-a^{-1}q\\ ap - bAq - A^{-1}q\end{pmatrix}_{(a,b)=f_s(p,q)}, \tag{4.1}$$

$$(a,b) = f_s(p,q) = \left(\frac{\langle A^{-1}q,q\rangle}{2\langle p,q\rangle - 1}, \frac{\langle A^{-1}q,q\rangle}{(2\langle p,q\rangle - 1)\langle Aq,q\rangle}\right). \tag{4.2}$$

The jth component of equation (4.1) reads

$$E\begin{pmatrix}p_j\\ q_j\end{pmatrix} = \begin{pmatrix}0 & -a^{-1}\\ a & -b\alpha_j - \alpha_j^{-1}\end{pmatrix}\begin{pmatrix}p_j\\ q_j\end{pmatrix},$$

which is equation (1.4) with $\lambda = \alpha_j, \chi = (p_j, q_j)^T$ and a, b expressed as in the constraint (4.2). Through direct calculations we see that S preserves $\mathrm{d}p \wedge \mathrm{d}q$, thus is symplectic [24–26]. Besides, the matrix U and the Lax–Moser matrix V_λ commute

$$(EV_\lambda)U - UV_\lambda = 0, \tag{4.3}$$

where E is the shift operator of the discrete flow S^n generated by S. Since $\det U = 1$, immediately we have $E \det V_\lambda = V_\lambda$, i.e. $EF_\lambda = F_\lambda$. Thus F_λ, and hence H_λ, F_k and H_k, are integrals of S.

Proposition 4.1. *The map S is symplectic and completely integrable in Liouville sense.*

The straightening out of the discrete S-flow in the Jacobi variety $J(\Gamma)$ is obtained similarly, as in [17–19]. Consider the fundamental solution matrix $M(k)$ of equation (1.4)

$$M(k+1) = U_k M(k), \qquad M(0) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}; \qquad M(K) = \begin{pmatrix}p^{(1)}(k,\lambda) & p^{(2)}(k,\lambda)\\ q^{(1)}(k,\lambda) & q^{(2)}(k,\lambda)\end{pmatrix}. \tag{4.4}$$

Evidently $M(k) = U_{k-1}U_{k-2}\cdots U_0$, $\det M(k) = 1$. Under the constraint (4.2), we have

$$V_\lambda(k)M(k) = M(k)V_\lambda(0). \tag{4.5}$$

By induction we obtain

$$M(k) = \lambda^{-k}\begin{pmatrix}\left(\tilde{M}_k^{11} + O(\lambda^2)\right)\lambda^2 & \left(\tilde{M}_k^{12} + O(\lambda^2)\right)\lambda\\ \left(\tilde{M}_k^{21} + O(\lambda^2)\right)\lambda & \tilde{M}_k^{22} + O(\lambda^2)\end{pmatrix}, \qquad (\lambda \to 0), \tag{4.6}$$

$$M(k) = \lambda^{k}\begin{pmatrix}\left(M_k^{11} + O(\lambda^{-2})\right)\lambda^{-2} & \left(M_k^{12} + O(\lambda^{-2})\right)\lambda^{-1}\\ \left(M_k^{21} + O(\lambda^{-2})\right)\lambda^{-1} & M_k^{22} + O(\lambda^{-2})\end{pmatrix}, \qquad (\lambda \to \infty). \tag{4.7}$$

$\lambda^{k-2}p^{(1)}(k,\lambda), \lambda^{k-1}p^{(2)}(k,\lambda), \lambda^{k-1}q^{(1)}(k,\lambda)and\lambda^k q^{(2)}(k,\lambda)$ are polynomials of $\zeta = \lambda^2$ with degrees $k-2, k-1, k-1, k$, respectively. By equation (4.3), the solution space $\mathcal{E}_\lambda$ of $E\chi = U\chi$ is invariant under the action of V_λ. It has a eigenvalue $\rho^\pm = \pm\sqrt{-F_\lambda}$ with the associated eigenvector in $\mathcal{E}_\lambda$ as

$$\chi^\pm(k) = \begin{pmatrix}p^\pm(k,\lambda)\\ q^\pm(k,\lambda)\end{pmatrix} = \begin{pmatrix}p^{(1)}(k,\lambda) + d^\pm p^{(2)}(k,\lambda)\\ q^{(1)}(k,\lambda) + d^\pm q^{(2)}(k,\lambda)\end{pmatrix}. \tag{4.8}$$

Let $k = 0$ in $[V_\lambda(k) - \rho^\pm]\chi^\pm(k) = 0$. We have

$$d^\pm = -\frac{V_\lambda^{11}(0) - \rho^\pm}{V_\lambda^{12}(0)} = \frac{V_\lambda^{21}(0)}{V_\lambda^{11}(0) + \rho^\pm}. \tag{4.9}$$

By using equations (4.3) and (4.4), in a similar way as in [17–19], we obtain the discrete version of Dubrovin–Novikov formula

$$p^{+}(k,\lambda)p^{-}(k,\lambda)=\prod_{i=1}^{g}\frac{\zeta-\mu_i^2(k)}{\zeta-\mu_i^2(0)}. \tag{4.10}$$

Resorting to these results and equations (4.6)–(4.7), we have the asymptotic behaviors

$$\lambda^k p^{-}(k,\lambda)=\frac{4H_0\tilde{M}_k^{12}}{\langle A^{-1}p,p\rangle|_{k=0}}+O(\lambda^2),\qquad \lambda^k p^{+}(k,\lambda)=\left(\frac{\langle A^{-1}p,p\rangle|_k}{4H_0\tilde{M}_k^{12}}+O(\lambda^2)\right)\lambda^{2k}, \tag{4.11}$$

$$\lambda^k p^{-}(k,\lambda)=\left(\frac{M_k^{12}}{\langle Ap,p\rangle|_{k=0}}+O(\lambda^{-2})\right)\lambda^{2k},\qquad \lambda^k p^{+}(k,\lambda)=\frac{\langle Ap,p\rangle|_{k=0}}{M_k^{12}}+O(\lambda^{-2}), \tag{4.12}$$

for $\lambda\to 0$, $\lambda\to\infty$, respectively.

Proposition 4.2. *The Abel–Jacobi variable ϕ straightens out the symplectic flow S^k*

$$\phi(k)\equiv k\Omega_S+\phi(0),\qquad (\mathrm{mod}\,\mathcal{T}); \tag{4.13}$$

$$\Omega_s\equiv\int_{0_2}^{\infty_1}\omega\equiv\int_{\infty_2}^{0_1}\omega,\qquad (\mathrm{mod}\,\mathcal{T}). \tag{4.14}$$

Proof. $\lambda^k p^{+}(k,\lambda)$, $\lambda^k p^{-}(k,\lambda)$ are polynomials of ζ,ζ^{-1}, which are values of a well-defined meromorphic function $\psi(k,P)$ in the two sheets of Γ, respectively. It has simple zeros and simple poles at $P\big(\mu_i^2(k)\big)$, $P\big(\mu_i^2(0)\big)$, respectively, a kth order pole at ∞_1 and a kth order zero at 0_2. By a method due to Toda [17–19, 27], we get

$$\sum_{i=1}^{g}\int_{P(\mu_i^2(0))}^{P(\mu_i^2(k))}\omega+k\int_{\infty_1}^{0_2}\omega\equiv 0,\qquad (\mathrm{mod}\,\mathcal{T}).$$

This implies equation (4.13). By using the map $\tau: P=(\zeta,\xi)\to P^{-}=(\zeta,-\xi)$ with the property $\tau^2=id|_\Gamma$ and $\tau^*\omega=-\omega$, we obtain

$$\Omega_s=\int_{0_2}^{\infty_1}\omega=\int_{0_2}^{\infty_1}\tau^2\omega=\int_{0_1}^{\infty_2}\tau^*\omega=-\int_{0_1}^{\infty_2}\omega,\qquad (\mathrm{mod}\,\mathcal{T}).$$

□

Lemma 4.3.

$$E\widetilde{m}(\zeta)=\widetilde{n}(\zeta)\qquad\text{and}\qquad E\sum_{i=1}^{g}\mu_i^{2k}=\sum_{i=1}^{g}\nu_i^{2k},\qquad k\in\mathbb{Z}.$$

Proof. By equation (4.1), $\overline{p}=-a^{-1}q$. The proof is completed by substituting the following expressions into equation (3.20)

$$Q_\lambda(A\overline{p},\overline{p})=a^{-2}Q_\lambda(Aq,q),\qquad \langle A\overline{p},\overline{p}\rangle=a^{-2}\langle Aq,q\rangle.$$

□

5. Solution of the mKP equation (1.6)

(i) *The Abel–Jacobi solutions*

Let $(u, v) = f^+(p, q)$ and (p, q) be solution of (H_1). Consider

$$G_\lambda = (-\lambda Q_\lambda(Ap, p), \lambda Q_\lambda(Aq, q), \lambda^{-1}[1/2 + Q_\lambda(A^2 p, q)])^T. \tag{5.1}$$

By equations (2.1), (2.2) and (3.1), we have $V_\lambda = \sigma_\lambda(G_\lambda)$ and $(K^+ - \lambda^2 J^+)G_\lambda = 0$. In the same way as in [16], we obtain

$$\partial_{t_\lambda}(u, v, 1/2)^T = 2\lambda J^+ G_\lambda; \tag{5.2}$$

$$\lambda G_\lambda = -4H_\lambda g_\lambda. \tag{5.3}$$

After changing $\mathrm{d}/\mathrm{d}t_\lambda$ into $\mathrm{d}/\mathrm{d}\tau_\lambda$ by equation (3.10), we get

$$\frac{\mathrm{d}}{\mathrm{d}\tau_\lambda}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} (J^+ g_\lambda)^1 \\ (J^+ g_\lambda)^2 \end{pmatrix} = \sum_{k=1}^{\infty} X_k \lambda^{-2k}.$$

Proposition 5.1. *Let $(p(x, \tau_k), q(x, \tau_k))$ be compatible solution of $(H_1), (H_k)$. Then $(u, v) = f^+(p, q)$ solves the KN equation (2.5).*

Proposition 5.2. *Let $(p(x, \tau_2, \tau_3), q(x, \tau_2, \tau_3))$ be compatible solution of $(H_1), (H_2, (H_3)$. Then the mKP equation (1.6) has a special solution*

$$w(x, \tau_2, \tau_3) = uv = -\langle Ap, p\rangle\langle Aq, q\rangle. \tag{5.4}$$

By equation (3.28), we have the solutions expressed in the Abel–Jacobi variables

$$KN(2.5): \qquad \phi = x\Omega_1 + \tau_k\Omega_k + \phi_0, \tag{5.5}$$

$$\text{equation} \quad (2.14): \qquad \phi = x\Omega_1 + n\Omega_S + \phi_0, \tag{5.6}$$

$$mKP(1.6): \qquad \phi = x\Omega_1 + \tau_2\Omega_2 + \tau_3\Omega_3 + \phi_0. \tag{5.7}$$

(ii) *Trace formulas*

According to Riemann's theorem, by equation (3.26), there exists a constant vector K such that $\theta(\mathcal{A}(P) - \phi - K)$ has simple zeros at $P(\mu_j^2)$, $1 \leqslant j \leqslant g$. By equation (3.29), in the local coordinate $z = \zeta^{-1}$ near $P = \infty_l$, we obtain

$$\mathcal{A}(P(\zeta)) = \int_{P_0}^{P(\zeta)} \omega = (-1)^{l-1}\sum_{k=1}^{\infty}\frac{1}{k}\Omega_k z^k - \eta_{\infty_l}, \qquad \eta_{\infty_l} = \int_{\infty_l}^{P_0}\omega, \tag{5.8}$$

In quite a similar way as in [16] we have

$$\begin{gathered} \sum_{i=1}^{g}\mu_i^{2k} = I_k(\Gamma) - \sum_{l=1}^{2}\operatorname*{Res}_{\infty_l}\zeta^k d\ln\theta(\mathcal{A}(P) - \phi - K), \qquad I_k(\Gamma) = \sum_{i=1}^{g}\int_{a_i}\zeta^k\omega_i, \\ \sum_{i=1}^{g}\mu_i^2 = I_1(\Gamma) + \Omega_1^i\partial_i\ln\frac{\theta_1}{\theta_2}, \qquad \sum_{i=1}^{g}\mu_i^4 = I_2(\Gamma) + \Omega_2^i\partial_i\ln\frac{\theta_1}{\theta_2} - \Omega_1^i\Omega_1^j\partial_{ij}^2\ln\theta_1\theta_2, \end{gathered} \tag{5.9}$$

where $\theta_l = \theta(\phi + K + \eta_{\infty_l})$ and ∂_i designates the partial derivative with respect to the ith argument of the theta function. Let ϕ be the Abel–Jacobi solution: $\phi = \phi_0 + \sum\tau_k\Omega_k$. Then $\Omega_k^i\partial_i = \partial_{\tau_k}$. With the zeros λ_i^2 of $R(\zeta)$ defined as in equation (3.16), consider

$$S_k = \frac{1}{2}\sum_{i=1}^{2N}\lambda_i^{2k} - \sum_{i=1}^{g}\mu_i^{2k}, \qquad \tilde{S}_k = \frac{1}{2}\sum_{i=1}^{2N}\lambda_i^{2k} - \sum_{i=1}^{g}\nu_i^{2k}. \tag{5.10}$$

Proposition 5.3. *Let* $D_k = \sum_{i=1}^{2N}\left(\lambda_i^{2k}/2\right) - I_k(\Gamma)$. *Then*

$$\begin{aligned} S_1 &= -\partial_{\tau_1}\ln\frac{\theta_1}{\theta_2} + D_1, & S_2 &= -\partial_{\tau_2}\ln\frac{\theta_1}{\theta_2} + \partial_{\tau_1}\ln\theta_1\theta_2 + D_2,\\ \tilde S_1 &= -\partial_{\tau_1}\ln\frac{\tilde\theta_1}{\tilde\theta_2} + D_1, & \tilde S_2 &= -\partial_{\tau_2}\ln\frac{\tilde\theta_1}{\tilde\theta_2} + \partial_{\tau_1}\ln\tilde\theta_1\tilde\theta_2 + D_2, \end{aligned} \tag{5.11}$$

where $\theta_l = \theta\left(\sum\tau_k\Omega_k + \phi_0 + K + \eta_{\infty_l}\right)$, $\tilde\theta_l = \theta\left(\sum\tau_k\Omega_k + \phi_0 + K + \eta_{\infty_l} + \Omega_S\right)$.

Proof. From equation (5.9) we have the first two formulae. According to Lemma 4.3, the latter two formulae are obtained by exerting the operator E upon the former ones. □

(iii) *The Jacobi inversion*

The direct relations between $(u, v) = f^+(p, q)$ and the elliptic coordinates are given by

$$+\frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}\tau_\lambda} + 2g_\lambda^3 = \frac{\prod_{i=1}^{g}\left(1-\mu_i^2 z\right)}{\sqrt{\prod_{i=1}^{2N}\left(1-\lambda_i^2 z\right)}}, \qquad -\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}\tau_\lambda} + 2g_\lambda^3 = \frac{\prod_{i=1}^{g}\left(1-\nu_i^2 z\right)}{\sqrt{\prod_{i=1}^{2N}\left(1-\lambda_i^2 z\right)}}. \tag{5.12}$$

They are proved by substituting equation (5.1) into (5.2), and taking into account of equation (5.3) and (3.20). Through power series expansions we get

$$kT_k = S_k + \sum_{i+j=k;\,i,j\geqslant 1} T_i S_j, \qquad k\tilde T_k = \tilde S_k + \sum_{i+j=k;\,i,j\geqslant 1} \tilde T_i \tilde S_j, \tag{5.13}$$

where $T_k = \partial_{\tau_k}\ln u + 2g_k^3$, $\tilde T_k = -\partial_{\tau_k}\ln v + 2g_k^3$. In particular,

$$\begin{aligned} S_1 &= T_1, & S_2 &= 2T_2 - T_1^2;\\ \tilde S_1 &= \tilde T_1, & \tilde S_2 &= 2\tilde T_2 - \tilde T_1^2. \end{aligned} \tag{5.14}$$

Lemma 5.4. *Let* $(u, v) = f^+(p, q)$. *Then*

$$\partial_{\tau_\lambda} g_\mu^3 = \partial_{\tau_\mu} g_\lambda^3, \qquad \forall\lambda, \mu \in \mathbb{C}; \tag{5.15}$$

$$\partial_{\tau_i} g_j^3 = \partial_{\tau_j} g_i^3, \qquad \forall i, j = 1, 2, \ldots. \tag{5.16}$$

Proof. From equation (3.5) we have

$$\frac{\mathrm{d}V_\lambda^{11}}{\mathrm{d}t_\lambda} = \frac{2\lambda\mu}{\lambda^2-\mu^2}\begin{vmatrix} V_\lambda^{12} & V_\mu^{12}\\ V_\lambda^{21} & V_\mu^{21}\end{vmatrix}.$$

By equations (5.3), (5.1), (3.1) and (3.18), this equation is transformed into

$$\frac{\mathrm{d}g_\mu^3}{\mathrm{d}\tau_\lambda} = \frac{1}{\lambda^2-\mu^2}\begin{vmatrix} g_\lambda^1 & g_\mu^1\\ g_\lambda^2 & g_\mu^2\end{vmatrix}.$$

The symmetry with respect to λ, μ proves equation (5.15). The power series expansion gives rise to equation (5.16). □

Lemma 5.5.

$$\begin{aligned} &+uv_x + u^2v^2 = \partial_x^2\ln\theta_1 + \tfrac{1}{2}(D_2 - N_2'),\\ &-u_xv + u^2v^2 = \partial_x^2\ln\tilde\theta_2 + \tfrac{1}{2}(D_2 - N_2''),\\ &(uv)_x = \partial_x^2\ln(\theta_1/\tilde\theta_2) + \tfrac{1}{2}(N_2'' - N_2'). \end{aligned} \tag{5.17}$$

Proof. The first two members of equation (5.14) read

$$\partial_x \ln u + 2g_1^3 = -\partial_x \ln(\theta_1/\theta_2) + D_1; \tag{5.18}$$

$$2\partial_{\tau_2} \ln u + 4g_1^3 - T_1^2 = -\left(\partial_{\tau_2} + \partial_x^2\right) \ln(\theta_1/\theta_2) + 2\partial_x^2 \ln\theta_2 + D_2. \tag{5.19}$$

Differentiate equation (5.18) with respect to τ_2

$$\partial_{\tau_2}\partial_x \ln u + 2\partial_{\tau_2} g_1^3 = -\partial_{\tau_2}\partial_x \ln(\theta_1/\theta_2)$$

By equation (5.16), $\partial_{\tau_2} g_1^3 = \partial_x g_2^3$, since $x = \tau_1$. Thus we have

$$\partial_{\tau_2} \ln u + 2g_2^3 = -\partial_{\tau_2} \ln(\theta_1/\theta_2) + N_2', \tag{5.20}$$

where N_2' is independent of x. By using this equation and equation (5.18) to cancel the extra terms in equation (5.19), we obtain the first formula of equation (5.17). Similarly, from the latter two members in equation (5.14)

$$-\partial_x \ln v + 2g_1^3 = -\partial_x \ln(\tilde\theta_1/\tilde\theta_2) + D_1,$$

$$-2\partial_{\tau_2} \ln v + 4g_1^3 - \tilde T_1^2 = -\left(\partial_{\tau_2} - \partial_x^2\right) \ln(\tilde\theta_1/\tilde\theta_2) + 2\partial_x^2 \ln\tilde\theta_2 + D_2,$$

we have

$$-\partial_{\tau_2} \ln v + 2g_2^3 = -\partial_{\tau_2} \ln(\tilde\theta_1/\tilde\theta_2) + N_2'',$$

where N_2'' is independent of x. By canceling the extra terms we have the second formula in equation (5.17). The third one is a corollary of the first two ones. □

Proposition 5.6. *The mKP equation (1.6) has a finite genus solution:*

$$w(x,\tau_2,\tau_3) = \partial_x \ln \frac{\theta(x\Omega_1 + \tau_2\Omega_2 + \tau_3\Omega_3 + \delta_{\infty_1})}{\theta\left(x\Omega_1 + \tau_2\Omega_2 + \tau_3\Omega_3 + \delta_{\infty_2} + \Omega_s\right)} + N_2 x + D_+, \tag{5.21}$$

where $\delta_{\infty_l} = \phi_0 + K + \eta_{\infty_l}$*;* $N_2 = (N_2'' - N_2')/2$ *and* D_+ *are independent of* x*.*

6. Solution of the (2+1) d-Toda equation

(i) *Abel–Jacobi solutions containing* H_{-k}*-flow*

Let $(r,s) = f_-(p,q)$ and (p,q) be a solution of (H_{-1}). A straightforward calculation confirms that the equation $(K^- - \lambda^{-2}J^-)\tilde G_\lambda = 0$ has a solution $\tilde G_\lambda$:

$$\begin{aligned} &\tilde G_\lambda^1 = Q_\lambda(Ap,p), \qquad \tilde G_\lambda^2 = -Q_\lambda(Aq,q),\\ &\tilde G_\lambda^3 = -\left[\tfrac12 + Q_\lambda(A^2p,q)\right] + \lambda^2[sQ_\lambda(Ap,p) - rQ_\lambda(Aq,q)]. \end{aligned} \tag{6.1}$$

Similar calculations give rise to

$$\frac{\mathrm d}{\mathrm dt_\lambda}\begin{pmatrix} r\\ s\end{pmatrix} = 2\begin{pmatrix} Q_\lambda(Ap,p)\\ Q_\lambda(Aq,q)\end{pmatrix}, \qquad \frac{\mathrm d}{\mathrm d\tau_{-k}}\begin{pmatrix} r\\ s\end{pmatrix} = X_{-k}|_{(r,s)=f^-(p,q)} \tag{6.2}$$

Proposition 6.1. *Let* $p(y,\tau_{-k}), q(y,\tau_{-k})$ *be a compatible solution of* $(H_{-1}), (H_{-k})$*. Then the nKN equation (2.11) has a solution:* $(r,s) = f^-(p,q)$*.*

Proposition 6.2. *Let* $p(y,\tau_{-2},\tau_{-3}), q(y,\tau_{-2},\tau_{-3})$ *be a compatible solution of* $(H_{-1}), (H_{-2})$*and*(H_{-3})*. Then the mKP equation (1.7) has a solution*

$$\tilde w = rs = -\frac{\langle A^{-1}p,p\rangle\langle A^{-1}q,q\rangle}{(2\langle p,q\rangle - 1)^2}. \tag{6.3}$$

Immediately we have the Abel–Jacobi solutions for a series of integrable equations

$$nKN\ (2.11):\quad \phi = y\Omega_{-1} + \tau_{-k}\Omega_{-k} + \phi_0; \tag{6.4}$$

$$mKP\ (1.7):\quad \phi = y\Omega_{-1} + \tau_{-2}\Omega_{-2} + \tau_{-3}\Omega_{-3} + \phi_0; \tag{6.5}$$

$$(2+1)\ \text{dToda}\ (1.1):\quad \phi = x\Omega_1 + y\Omega_{-1} + n\Omega_S + \phi_0; \tag{6.6}$$

$$\text{equation}\ (2.15):\quad \phi = y\Omega_{-1} + n\Omega_S + \phi_0; \tag{6.7}$$

$$\text{equation}\ (2.13):\quad \phi = x\Omega_1 + y\Omega_{-1} + \phi_0. \tag{6.8}$$

(ii) *Trace formula for negative powers*

By equation (3.30), we have the expansion near $P = 0_l$

$$\mathcal{A}(P(\zeta)) = \int_{P_0}^{P(\zeta)} \omega = (-1)^l \sum_{k=1}^{\infty} \frac{1}{k}\Omega_{-k}\zeta^k - \eta_{0_l}, \qquad \eta_{0_l} = \int_{0_l}^{P_0} \omega. \tag{6.9}$$

Similar considerations yield

$$\sum_{i=1}^{g} \mu_i^{-2k} = I_{-k}(\Gamma) - \sum_{l=1}^{2} \operatorname*{Res}_{0_l} \zeta^{-k} d \ln \theta(\mathcal{A}(P) - \phi - K), \quad I_{-k}(\Gamma) = \sum_{i=1}^{g} \int_{a_i} \zeta^{-k}\omega_i, \tag{6.10}$$

$$\begin{aligned}
&\sum_{i=1}^{g} \mu_i^{-2} = I_{-1}(\Gamma) - \Omega_{-1}^i \partial_i \ln \frac{\theta_1^0}{\theta_2^0};\\
&\sum_{i=1}^{g} \mu_i^{-4} = I_{-2}(\Gamma) - \Omega_{-2}^i \partial_i \ln \frac{\theta_1^0}{\theta_2^0} - \Omega_{-1}^i \Omega_{-1}^j \partial_{ij}^2 \ln \theta_1^0\theta_2^0;\\
&\sum_{i=1}^{g} \mu_i^{-6} = I_{-3}(\Gamma) - \left(\Omega_{-3}^i \partial_i + \frac{1}{2}\Omega_{-1}^i\Omega_{-1}^j\Omega_{-1}^k \partial_{ijk}^3\right) \ln \frac{\theta_1^0}{\theta_2^0} - \frac{3}{2}\Omega_{-2}^i\Omega_{-1}^j \partial_{ij}^2 \ln \theta_1^0\theta_2^0.
\end{aligned} \tag{6.11}$$

where $\theta_l^0 = \theta(\phi + K + \eta_{0_l})$. For the Abel–Jacobi solution $\phi = \phi_0 + \sum \tau_k\Omega_k + \sum \tau_{-k}\Omega_{-k} + n\Omega_S$, we have $\Omega_{-k}^i \partial_i = \partial_{\tau_{-k}}$. Thus

$$\begin{aligned}
&\sum_{i=1}^{g} \mu_i^{-2} = I_{-1}(\Gamma) - \partial_y \ln \frac{\theta_1^0}{\theta_2^0};\\
&\sum_{i=1}^{g} \mu_i^{-4} = I_{-2}(\Gamma) - \partial_{\tau_{-2}} \ln \frac{\theta_1^0}{\theta_2^0} - \partial_y^2 \ln \theta_1^0\theta_2^0;\\
&\sum_{i=1}^{g} \mu_i^{-6} = I_{-3}(\Gamma) - \left(\partial_{\tau_{-3}} + \frac{1}{2}\partial_y^3\right) \ln \frac{\theta_1^0}{\theta_2^0} - \frac{3}{2}\partial_{\tau_{-2}}\partial_y \ln \theta_1^0\theta_2^0.
\end{aligned} \tag{6.12}$$

(iii) *Inversion in the negative case*

Proposition 6.3. *Let* $(r, s) = f^-(p, q)$. *Then*

$$\begin{aligned}
&-\frac{1}{r}\frac{\mathrm{d}r}{\mathrm{d}\tau_\lambda} = \frac{\prod_{i=1}^{g}\left(1 - \mu_i^{-2}\zeta\right)}{\sqrt{\prod_{i=1}^{2N}\left(1 - \lambda_i^{-2}\zeta\right)}},\\
&+\frac{1}{s}\frac{\mathrm{d}s}{\mathrm{d}\tau_\lambda} = \frac{\prod_{i=1}^{g}\left(1 - \nu_i^{-2}\zeta\right)}{\sqrt{\prod_{i=1}^{2N}\left(1 - \lambda_i^{-2}\zeta\right)}}.
\end{aligned} \tag{6.13}$$

Proof. By equation (3.3), the derivative of $(r, s) = f^-(p, q)$ with respect to t_λ is expressed as

$$\frac{\mathrm{d}r}{\mathrm{d}t_\lambda} = 2Q_\lambda(Ap, p), \qquad \frac{\mathrm{d}s}{\mathrm{d}t_\lambda} = 2Q_\lambda(Aq, q).$$

It is transformed into equation (6.13) by taking into account of the following four points: (a) the definition of f^- in equation (3.14); (b) the expression of H_0 in equation (3.8); (c) the relation given in equation (3.18) and (d) the definition of the elliptic variables given by equation (3.20). □

Expand equation (6.13) as $1 + \sum_{k=1}^{\infty} T_{-k}\zeta^k$, where $T_{-k} = -r_{\tau_{-k}}/r$. Consider its logarithm

$$\ln\left(1 + \sum_{k=1}^{\infty} T_{-k}\zeta^k\right) = \sum_{i=1}^{g} \ln\left(1 - \mu_i^{-2}\zeta\right) - \frac{1}{2}\sum_{i=1}^{2N} \ln\left(1 - \lambda_i^{-2}\zeta\right).$$

Differentiating with respect to ζ and comparing the coefficients, we obtain the recursive formula

$$kT_{-k} = S_{-k} + \sum_{i+j=k;i,j\geqslant 1} T_{-i}S_{-j}, \qquad S_{-k} \triangleq \frac{1}{2}\sum_{i=1}^{2N}\lambda_i^{-2k} - \sum_{i=1}^{g}\mu_i^{-2k}; \tag{6.14}$$

$$S_{-1} = T_{-1}, \qquad S_{-2} = 2T_{-2} - (T_{-1})^2, \qquad S_{-3} = 3T_{-3} - 3T_{-2}T_{-1} + (T_{-1})^3. \tag{6.15}$$

Substituting the trace formula in them, we have

Proposition 6.4.

$$\begin{aligned} &D_{-1} + \partial_y \ln\frac{\theta_1^0}{\theta_2^0} = -\frac{r_y}{r};\\ &D_{-2} + \partial_{\tau_{-2}} \ln\frac{\theta_1^0}{\theta_2^0} + \partial_y^2 \ln\theta_1^0\theta_2^0 = -2\frac{r_{\tau_{-2}}}{r} - \frac{r_y^2}{r^2};\\ &D_{-3} + \left(\partial_{\tau_{-3}} + \frac{1}{2}\partial_y^3\right)\ln\frac{\theta_1^0}{\theta_2^0} + \frac{3}{2}\partial_{\tau_{-2}}\partial_y \ln\theta_1^0\theta_2^0 = -3\frac{r_{\tau_{-3}}}{r} - 3\frac{r_{\tau_{-2}}r_y}{r^2} - \frac{r_y^3}{r^3}, \end{aligned} \tag{6.16}$$

where $D_{-k} \triangleq \sum_{i=1}^{2N}\left(\lambda_i^{-2k}/2\right) - I_{-k}(\Gamma)$, $\theta_l^0 = \theta\left(\phi + K + \eta_{0_l}\right)$, $\eta_{0_l} \triangleq \int_{0_l}^{P_0}\omega$, $l = 1, 2$.

Proposition 6.5. *Let* $\tilde\theta_l = \theta(\phi + K + \eta_{0_l} + \Omega_S)$, $(l = 1, 2)$. *Then*

$$\begin{aligned} &D_{-1} + \partial_y \ln\frac{\tilde\theta_1^0}{\tilde\theta_2^0} = \frac{s_y}{s};\\ &D_{-2} + \partial_{\tau_{-2}} \ln\frac{\tilde\theta_1^0}{\tilde\theta_2^0} + \partial_y^2 \ln\tilde\theta_1^0\tilde\theta_2^0 = 2\frac{s_{\tau_{-2}}}{s} - \frac{s_y^2}{s^2};\\ &D_{-3} + \left(\partial_{\tau_{-3}} + \frac{1}{2}\partial_y^3\right)\ln\frac{\tilde\theta_1^0}{\tilde\theta_2^0} + \frac{3}{2}\partial_{\tau_{-2}}\partial_y \ln\tilde\theta_1^0\tilde\theta_2^0 = 3\frac{s_{\tau_{-3}}}{s} - 3\frac{s_{\tau_{-2}}s_y}{s^2} + \frac{s_y^3}{s^3}. \end{aligned} \tag{6.17}$$

Proposition 6.6. *The (2+1) d-Toda equation (1.1) has a finite genus solution:*

$$\varphi_n(x, y) = \ln\frac{\theta\left(x\Omega_1 + y\Omega_{-1} + n\Omega_S + \delta_{0_2}\right)\theta\left(x\Omega_1 + (n+1)\Omega_S + \delta_{\infty_2}\right)}{\theta\left(x\Omega_1 + y\Omega_{-1} + (n+1)\Omega_S + \delta_{\infty_2}\right)\theta\left(x\Omega_1 + n\Omega_S + \delta_{0_2}\right)} + D_{-1}y + \varphi_n(x, 0), \tag{6.18}$$

where $\delta_{0_2} = \phi_0 + K + \eta_{0_2}$, $\delta_{\infty_2} = \phi_0 + K + \eta_{\infty_2}$. D_{-1} *is independent of* x, y *and* n.

Proof. By the notations in lemma (2.7)–(2.8), we have $\varphi_n = \ln a_n = \ln s$. From equation (6.17) we get

$$\partial_y \varphi_n = \partial_y \ln\left(\tilde{\theta}_1^0 / \tilde{\theta}_2^0\right) + D_{-1},$$
$$\varphi_n(x, y) = \ln\left(\tilde{\theta}_1^0 / \tilde{\theta}_2^0\right) - \ln\left(\tilde{\theta}_1^0 / \tilde{\theta}_2^0\right)\big|_{y=0} + D_{-1} y + \varphi_n(x, 0),$$

with the following expressions, where $\eta_{0_1} + \Omega_S = \eta_{\infty_2}$ by equation (4.14) is used:

$$\begin{aligned}\tilde{\theta}_1^0 &= \theta\left(x\Omega_1 + y\Omega_{-1} + n\Omega_S + \phi_0 + K + \eta_{0_1} + \Omega_S\right)\\ &= \theta\left(x\Omega_1 + y\Omega_{-1} + n\Omega_S + \delta_{\infty_2}\right),\\ \tilde{\theta}_2^0 &= \theta\left(x\Omega_1 + y\Omega_{-1} + n\Omega_S + \phi_0 + K + \eta_{0_2} + \Omega_S\right)\\ &= \theta\left(x\Omega_1 + y\Omega_{-1} + (n+1)\Omega_S + \delta_{0_2}\right).\end{aligned}$$

□

7. Solution of the nKN equation and mKP (1.7)

From the first members of equation (6.16)–(6.17) we obtain

$$-\ln r = \ln\left(\theta_1^0 / \theta_2^0\right) + D_{-1} y + c', \qquad (7.1)$$

$$+\ln s = \ln\left(\tilde{\theta}_1^0 / \tilde{\theta}_2^0\right) + D_{-1} y + c'', \qquad (7.2)$$

$$-\partial_{\tau_{-k}} \ln r = \partial_{\tau_{-k}} \ln\left(\theta_1^0 / \theta_2^0\right) + D_{-1} y + N'_{-k}, \qquad (7.3)$$

$$+\partial_{\tau_{-k}} \ln s = \partial_{\tau_{-k}} \ln\left(\tilde{\theta}_1^0 / \tilde{\theta}_2^0\right) + D_{-1} y + N''_{-k}, \qquad (7.4)$$

where $c', c'', N'_{-k}, N''_{-k}$ are independent of the argument y.

Lemma 7.1.

$$-r s_y - r^2 s^2 = \partial_y^2 \ln \theta_1^0 + (D_{-2} - N'_{-2})/2, \qquad (7.5)$$

$$+r_y s - r^2 s^2 = \partial_y^2 \ln \tilde{\theta}_2^0 + (D_{-2} - N''_{-2})/2, \qquad (7.6)$$

$$(rs)_y = \partial_y^2 \ln\left(\tilde{\theta}_2^0 / \theta_1^0\right) + (N'_{-2} - N''_{-2})/2. \qquad (7.7)$$

Proof. By equation (6.16) and (7.3) we have

$$-2\partial_{\tau_{-2}} \ln r - \frac{r_y^2}{r^2} = \left(\partial_{\tau_{-2}} - \partial_y^2\right) \ln \frac{\theta_1^0}{\theta_2^0} + 2\partial_y^2 \ln \theta_1^0 + D_{-2},$$

$$-\left(\partial_{\tau_{-2}} - \partial_y^2\right) \ln r = \left(\partial_{\tau_{-2}} - \partial_y^2\right) \ln \frac{\theta_1^0}{\theta_2^0} - N'_{-2}$$

By canceling the first terms in the right-hand side, and using equation (2.12) to calculate $\partial r / \partial \tau_{-2}$, we obtain equation (7.5). A similar calculation for the equations of s leads to the proof of equation (7.6). The third equality is a corollary of the first two ones.

A little more complicated calculation in the case $k = 3$ gives rise to □

Lemma 7.2.

$$-r s_{yy} + r_y s_y - 2r^2 s s_y = \partial_{\tau_{-2}} \partial_y \ln \theta_1^0 + (D_{-3} - N'_{-3})/3, \qquad (7.8)$$

$$-r_{yy} s + r_y s_y + 2r s^2 r_y = \partial_{\tau_{-2}} \partial_y \ln \tilde{\theta}_2^0 + (D_{-3} - N''_{-3})/3, \qquad (7.9)$$

$$(rs)_{\tau_{-2}} = \partial_{\tau_{-2}}\partial_y \ln\left(\tilde{\theta}_2^0/\theta_1^0\right) + (N'_{-3} - N''_{-3})/3. \tag{7.10}$$

Lemma 7.3. *N'_{-2}, N''_{-2} are independent of the argument τ_{-2}.*

Proof. Differentiate equation (7.5) and (7.8) with respect to τ_{-2}, y, respectively. Direct calculations gives

$$\partial_{\tau_{-2}}(rs_y + r^2s^2) = rs_{yyy} - r_{yy}s_y + 2r^2ss_{yy} + 4rsr_ys_y + 4r^2s_y^2 = \partial_y(rs_{yy} - r_ys_y + 2r^2ss_y).$$

Thus

$$\partial_{\tau_{-2}}(N'_{-2}/2) = \partial_y(N'_{-3}/3) = 0,$$

since N'_{-3} is independent of y. This proves the independence of N'_{-2} with respect to τ_{-2}. From equations (7.5) and (7.8) we have

$$\tfrac{1}{2}\partial_{\tau_{-2}}(N'_{-2} - N''_{-2}) = \tfrac{1}{3}\partial_y(N'_{-3} - N''_{-3}) = 0.$$

Hence N''_{-2} is independent of τ_{-2}. □

Proposition 7.4. *The nKN equation (1.5) has a finite genus solution:*

$$\begin{aligned} r(y,\tau_{-2}) &= r(0,0)\mathrm{e}^{-D_{-1}y - N'_{-2}\tau_{-2}} \frac{\theta\left(y\Omega_{-1} + \tau_{-2}\Omega_{-2} + \delta_{0_2}\right)\theta(\delta_{0_1})}{\theta\left(y\Omega_{-1} + \tau_{-2}\Omega_{-2} + \delta_{0_1}\right)\theta\left(\delta_{0_2}\right)}, \\ s(y,\tau_{-2}) &= s(0,0)\mathrm{e}^{+D_{-1}y + N''_{-2}\tau_{-2}} \frac{\theta\left(y\Omega_{-1} + \tau_{-2}\Omega_{-2} + \delta_{0_1} + \Omega_S\right)\theta\left(\delta_{0_2} + \Omega_S\right)}{\theta\left(y\Omega_{-1} + \tau_{-2}\Omega_{-2} + \delta_{0_2} + \Omega_S\right)\theta\left(\delta_{0_1} + \Omega_S\right)}, \end{aligned} \tag{7.11}$$

with $\delta_{0_l} = \phi_0 + K + \eta_{0_l}$, $l = 1, 2. D_{-1}$, N'_{-2}, N''_{-2} are independent of y and τ_{-2}.

Proof. By equations (6.17) and (7.3) we have

$$\partial_y \ln r = -\partial_y \ln\left(\theta_1^0/\theta_2^0\right) - D_{-1}, \qquad \partial_{\tau_{-2}} \ln r = -\partial_{\tau_{-2}} \ln\left(\theta_1^0/\theta_2^0\right) - N'_{-2},$$

Thus

$$\left(\ln r + \ln\left(\theta_1^0/\theta_2^0\right)\right)\Big|_{(0,0)}^{(y,\tau_{-2})} = -D_{-1}y - N'_{-2}\tau_{-2}.$$

This completes the proof of the first formula. The second one is proved in a similar way. □

Proposition 7.5. *The mKP equation (1.7) has a solution*

$$\tilde{w}(y,\tau_{-2},\tau_{-3}) = \partial_y \ln \frac{\theta\left(y\Omega_1 + \tau_{-2}\Omega_{-2} + \tau_{-3}\Omega_{-3} + \delta_{0_2} + \Omega_S\right)}{\theta\left(y\Omega_1 + \tau_{-2}\Omega_{-2} + \tau_{-3}\Omega_{-3} + \delta_{0_1}\right)} + N_{-2}y + D_{-}, \tag{7.12}$$

where $N_{-2} = (N'_{-2} - N''_{-2})/2$.

Proof. By equation (7.7). □

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